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APPLICATION OF PATCHING DIAGRAMS TO SOME QUESTIONS ABOUT PROJECTIVE MODULES

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1. Introduction

In this paper we prove some results for a type of cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

called patching diagrams (see Section 2 for definition) and give two applications.

Section 2 is essentially a résumé of facts intended for later use.

A natural question in the context of a patching diagram is the following: given projective modules P and Q over A such that they are isomorphic over B and C (after change of rings), when can we say that P and Q are isomorphic? In Section 3 we give some sufficient conditions for this to happen.

In Section 4 we give applications of the results of Section 3 to questions about projective modules over polynomial rings. The first application (Theorem 4.1) is an extension of a result of Kang who proved it under a finite normalization hypothesis on the ground ring. Our method is to reduce to Kang's case via a suitable patching diagram. As a second application (Theorem 4.2) we prove that for an analytically normal local domain R of dimension 2, the natural map $K_0(R[X_1, \ldots, X_n]) \to K_0(\hat{R}[X_1, \ldots, X_n])$ is an isomorphism.

For definitions and results pertaining to algebraic K-theory we refer the reader to [1].

Throughout this paper we shall be concerned only with commutative rings and finitely generated modules. The rings are not assumed to be noetherian in Sections 2 and 3.

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2. Patching diagrams

Let $f: A \to B$ be a homomorphism of rings and let s be an element of A such that

(i) s is a non-zero-divisor in A,

(ii) $f(s)$ is a non-zero-divisor in B,

(iii) f induces an isomorphism $A/sA \xrightarrow{\approx} B/f(s)B$.

The commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_s & \xrightarrow{f_s} & B_s \end{array} \qquad (2.1)$$

resulting from a situation as above will be called a *patching diagram,* borrowing the term from [7].

The conditions (i), (ii), and (iii) imply that f induces an isomorphism $A/s^nA \xrightarrow{\approx} B/f(s^n)B$ for all n, and that the diagram (2.1) is cartesian.

In this section and the next we shall work over a fixed patching diagram (2.1) *and results will be stated in the context of this diagram without always explicitly mentioning so.*

We begin by recalling a basic construction as given in Section 2 of [6]. Given a triple (P_1, σ, P_2), where P_1 and P_2 are finitely generated projective modules over A_s and B respectively, and $\sigma: (P_2)_s \to B_s \otimes_{A_s} P_1$ is a B_s-isomorphism, form the fibre product

$$M = M(P_1, \sigma, P_2) = \{(p_1, p_2) \in P_1 \times P_2 \mid 1 \otimes p_1 = \sigma(p_2/1)\}.$$

Then M has an obvious A-module structure. Moreover: (1) M is a finitely generated projective A-module; (2) the projections from M induce isomorphisms $M_s \to P_1$ and $B \otimes_A M \to P_2$; (3) if P is a finitely generated projective A-module and σ denotes the standard isomorphism $(B \otimes_A P)_s \to B_s \otimes_{A_s} P_s$, then P is canonically isomorphic to $M(P_s, \sigma, B \otimes_A P)$.

Proofs for the three statements above can be obtained by carrying over to our situation the proofs of similar statements in Section 2 of [6] noting two facts. The first is that Lemma 2.4 of [6] holds under the weaker hypothesis that the matrix $(a_{\alpha\beta})$ be a product of two matrices σ_1 and σ_2 such that σ_i is the image under j_i of an invertible matrix over A_i. The second fact is a result of Vorst which we quote:

Proposition 2.2. ([11], Lemma 2.4 (i)). *Given any $\alpha \in E_r(B_s)$, with $r \geq 3$, there exist $\beta \in \mathrm{Im}(E_r(A_s) \to E_r(B_s))$ and $\gamma \in \mathrm{Im}(E_r(B) \to E_r(B_s))$ such that $\alpha = \beta\gamma$.*

As a corollary to the discussion above, we have:

2.3. *If Q is a projective B-module such that Q_s is B_s-free, then Q is of the form $B \otimes_A P$ for some projective A-module P.*

This result also follows from Lemma 4 of [4].

At this point we make a convention that if P is a projective A-module, we shall *identify* $B_s \otimes_{A_s} P_s$ and $(B \otimes_A P)_s$ with $B_s \otimes_A P$. Accordingly P can be described as $M(P_s, 1, B \otimes_A P)$. If Q is another projective A-module and there are isomorphisms $\alpha : Q_s \to P_s$ and $\beta : B \otimes_A Q \to B \otimes_A P$, then it is easily seen that Q is isomorphic to $M(P_s, (1 \otimes \alpha)\beta_s^{-1}, B \otimes_A P)$. Further, Q is isomorphic to P if $1 \otimes \alpha = \beta_s$. Actually we have a necessary and sufficient condition for a module of the type $M(P_s, \sigma, B \otimes_A P)$ to be isomorphic to P and we record it as:

Remark 2.4. *Let σ be an automorphism of the B_s-module $B_s \otimes_A P$. Then $M(P_s, \sigma, B \otimes_A P)$ is isomorphic to P if and only if there exist automorphisms σ_1 of P_s and σ_2 of $B \otimes_A P$ such that $\sigma = (1 \otimes \sigma_1)(\sigma_2)_s$.*

The proof is not difficult.

Next we observe that for a patching diagram the associated diagram of the categories of projective modules is *E-surjective* in the sense of Definition 3.3 in Chapter VII of [1] (this can be checked using Proposition 2.2 stated earlier). As a special case of Theorem 4.3 of the same chapter we have:

Theorem 2.5. *The patching diagram* (2.1) *gives rise to exact sequences*

$$(E) \quad K_1(A) \to K_1(A_s) \oplus K_1(B) \to K_1(B_s) \to K_0(A) \to K_0(A_s) \oplus K_0(B) \to K_0(B_s)$$

and

$$(E') \quad 0 \to U(A) \to U(A_s) \oplus U(B) \to U(B_s) \to \mathrm{Pic}(A) \to \mathrm{Pic}(A_s) \oplus \mathrm{Pic}(B) \to \mathrm{Pic}(B_s).$$

Further, there is an epimorphism of exact sequences $\det : (E) \to (E')$.

For the definition of 'det' see Section 3 in Chapter IX of [1].

3. Factorization of automorphisms

Recall that in this section too we shall be working over a fixed patching diagram (2.1).

We saw in Remark 2.4 that the question of isomorphism of two projective A-modules amounts to finding factorizations of automorphisms over B_s into two parts which come from A_s and B. In this section we prove a result (Proposition 3.1) which describes a type of factorizable automorphisms and we derive from that a criterion for isomorphism of A-modules (Proposition 3.4). Proposition 3.1 is an analogue of Vorst's lemma, cited as Proposition 2.2, in a more general set up.

Proposition 3.1. *Let P be a finitely presented A-module and let σ be an automorphism of $B_s \otimes_A P$ which is a product of automorphisms of the type $1+c(1 \otimes \theta)$, where θ is an endomorphism of P_s with $\theta^2=0$ and $c \in B_s$. Then there exist automorphisms σ_1 of P_s and σ_2 of $B \otimes_A P$ such that $\sigma=(1 \otimes \sigma_1)(\sigma_2)_s$. Moreover, σ_1 and σ_2 can be chosen to be products of unipotent automorphisms of P_s and $B \otimes_A P$ respectively.*

Proof. Write $\sigma=\gamma_1 \cdots \gamma_m$, where $\gamma_i=1+c_i(1 \otimes \theta_i)$ with $c_i \in B_s$ and θ_i an endomorphism of P_s with $\theta_i^2=0$. Let $\delta_i=\gamma_{i+1} \cdots \gamma_m$ for $1 \leq i \leq m-1$ and $\delta_m=1$. Choose n sufficiently large so that there exists an endomorphism ϕ_i of $B \otimes_A P$ such that $\phi_i^2=0$ and $(\phi_i)_s=s^n \delta_i^{-1}(1 \otimes \theta_i)\delta_i$ (writing s instead of $f(s)$). It follows from the canonical isomorphism $\mathrm{End}_B(B \otimes_A P)_s \xrightarrow{\approx} \mathrm{End}_{B_s}(B_s \otimes_A P)$ that such an n exists. Let $c_i=b_i'/s^k$ with $b_i' \in B$ for $1 \leq i \leq m$. From the isomorphism $A/s^{n+k}A \to B/s^{n+k}B$ induced by f we derive that $b_i'=s^{n+k}b_i+f(a_i)$ with $b_i \in B$ and $a_i \in A$. So $c_i=s^n b_i/1+f(a_i/s^k)$. Since $\theta_i^2=0$ we have

$$\gamma_i=(1+f(a_i/s^k)(1 \otimes \theta_i))(1+(s^n b_i/1)(1 \otimes \theta_i)).$$

Denote the automorphism $(1+(a_1/s^k)\theta_1) \cdots (1+(a_m/s^k)\theta_m)$ of P_s by σ_1. It is easily checked that

$$\begin{aligned}\sigma &= (1 \otimes \sigma_1)\delta_m^{-1}(1+(s^n b_m/1)(1 \otimes \theta_m))\delta_m \cdots \delta_1^{-1}(1+(s^n b_1/1)(1 \otimes \theta_1))\delta_1 \\ &= (1 \otimes \sigma_1)(1+(b_m/1)(\phi_m)_s) \cdots (1+(b_1/1)(\phi_1)_s).\end{aligned}$$

Setting $\sigma_2=(1+b_m\phi_m) \cdots (1+b_1\phi_1)$ we are through.

Corollary 3.2. *The conclusion of the proposition above holds if P_s is free, say with a basis $\{e_1, \ldots, e_r\}$, and σ is in $E_r(B_s)$, considered as a matrix with respect to the basis $\{1 \otimes e_1, \ldots, 1 \otimes e_r\}$ of $B_s \otimes_{A_s} P$. Further, in this case σ_1 can be chosen so that it is in $E_r(A_s)$, viewed as a matrix with respect to the basis $\{e_1, \ldots, e_r\}$.*

In what follows we shall use Goldman's definition of determinant (see [2]). If a projective A-module P has a unimodular element, then $\det : \mathrm{End}_A(P) \to A$ is surjective. For, let $P=Ap \oplus P_1$ with p unimodular and let a be any element of A. Then the endomorphism of P which is identity on P_1 and sends p to ap has determinant a.

The next result is Lemma 4.1 of [3] stated in the context of a patching diagram.

Proposition 3.3. *Let P be a projective A-module of constant rank r, σ an automorphism of the B_s-module $B_s \otimes_A P$ and let Q denote the fibre product A-module $M(P_s, \sigma, B \otimes_A P)$. If $\wedge^r P \approx \wedge^r Q$, then there exist units $u \in A_s$ and $v \in B$ such that* $\det \sigma=(1 \otimes u)(v/1)$.

Proof. Let $[P,\sigma]$ denote the class of σ in $K_1(B_s)$ and let $[P]$ and $[Q]$ denote the classes of P and Q in $K_0(A)$. Consider the commutative diagram

$$\begin{array}{ccccc} & & K_1(B_s) & \xrightarrow{\partial} & K_0(A) \\ & & \downarrow \det & & \downarrow \det \\ U(A_s)\oplus U(B) & \xrightarrow{h} & U(B_s) & \xrightarrow{\partial} & \mathrm{Pic}(A) \end{array}$$

with exact row. We have $\det([P,\sigma])=\det\sigma$ and $(\det\circ\partial)([P,\sigma])=\det([Q]-[P])=0$ because $\bigwedge^r P\approx\bigwedge^r Q$. So $\det\sigma\in\mathrm{Im}\, h$.

Proposition 3.4. *Let* $\mathrm{SL}_r(B_s)=E_r(B_s)$ *for some r. Let P and Q be projective A-modules of rank r such that*

(i) $\bigwedge^r P\approx\bigwedge^r Q$,

(ii) P_s *and* Q_s *are free over* A_s,

(iii) $B\otimes_A P\approx B\otimes_A Q$ *and* $B\otimes_A Q$ *has a unimodular element.*

Then $P\approx Q$.

Proof. Let $\alpha: P_s\to Q_s$ and $\beta: B\otimes_A P\to B\otimes_A Q$ be isomorphisms over A_s and B respectively. By Proposition 3.3 $\det((1\otimes\alpha)\beta_s^{-1})=(1\otimes u)(v/1)$ with $u\in U(A_s)$ and $v\in U(B)$. Since Q_s and $B\otimes_A Q$ have unimodular elements, there are automorphisms γ of Q_s and δ of $B\otimes_A Q$ such that $\det\gamma=u$ and $\det\delta=v$. Replacing α by $\gamma^{-1}\alpha$ and β by $\delta\beta$ we may assume that $\det((1\otimes\alpha)\beta_s^{-1})=1$. Let $\{e_1,\dots,e_r\}$ be a basis of Q_s. Then $(1\otimes\alpha)\beta_s^{-1}$, regarded as a matrix with respect to the basis $\{1\otimes e_1,\dots,1\otimes e_r\}$ of $B_s\otimes_{A_s} Q_s$, belongs to $E_r(B_s)$. It follows from Remark 2.4 and Corollary 3.2 that $P\approx Q$.

4. Projective modules over polynomial and Laurent polynomial rings

Here we apply the results of the previous section to two situations (Theorems 4.1 and 4.2).

Recall that if R is artinian and A denotes the ring $R[X_1,\dots,X_n,Y_1^{\pm1},\dots,Y_m^{\pm1}]$, then projective A-modules of *constant* rank are free (use Corollary 1.4 of [10] after going modulo nilpotent elements). Further, it follows from Corollary 7.11 of [9] that $\mathrm{SL}_r(A)=E_r(A)$ for $r\geq 3$. Here again it suffices to do the checking modulo nilpotent elements.

Theorem 4.1. *Let R be a noetherian ring of dimension* 1, $A=R[X_1,\dots,X_n,Y_1^{\pm1},\dots,Y_m^{\pm1}]$ *and P a projective A-module. If rank* $P\geq 3$, *then P is a direct sum of a free A-module and a projective A-module of rank* ≤ 1.

If we assume R_{red} has finite normalization, then the theorem follows from the methods of Section 4 of [3]: the steps leading to a proof of Corollary 4.5 go through for the Laurent polynomial case as well.

Proof of Theorem 4.1. We may decompose R as a finite product of indecomposable rings, if necessary, and assume P to be of constant rank, say r. Let $\tilde{P}$ denote the A-module $\bigwedge^r P \otimes A^{r-1}$. We shall show that $P \approx \tilde{P}$. It is sufficient to do this modulo the nilradical of A. Therefore we assume A (equivalently R) to be reduced.

Let S denote the set of non-zero-divisors of R. Then R_S is a finite product of fields. So the projective A_S-modules P_S and $\tilde{P}_S$ are free. Choose $s \in S$ so that P_s and $\tilde{P}_s$ are free.

Let $\hat{R}$ denote the s-adic completion of R. We claim that $\hat{R}$ is semi-local. To see this first note that R_{1+sR} is semi-local because it is 1-dimensional and contains a non-zero-divisor (for instance s) in its Jacobson radical. Now $\hat{R}$ is the completion of R_{1+sR} and hence is semi-local. Since $\hat{R}_{red}$ has finite normalization, it follows from what we said after the statement of Theorem 4.1, that $\hat{A} \otimes_A P \approx \hat{A} \otimes_A \tilde{P}$, where $\hat{A}$ denotes $\hat{R} \otimes_R A$. Now $\hat{R}_s$ being artinian, we have $\mathrm{SL}_r(\hat{A}_s) = E_r(\hat{A}_s)$. Applying Proposition 3.4 to the patching diagram

$$\begin{array}{ccc} A & \longrightarrow & \hat{A} \\ \downarrow & & \downarrow \\ A_s & \longrightarrow & \hat{A}_s \end{array}$$

and the A-modules P and $\tilde{P}$ we get $P \approx \tilde{P}$.

Theorem 4.2. *Let R be an analytically normal local domain of dimension 2, let $A = R[X_1, \ldots, X_n]$ and let $\hat{A}$ denote $\hat{R}[X_1, \ldots, X_n]$. Then*

(i) *every projective $\hat{A}$-module is of the form $\hat{A} \otimes_A P$ for some projective A-module P,*

(ii) *a projective A-module P of rank ≥ 3 is free if $\hat{A} \otimes_A P$ is free.*

In particular, the canonical map $K_0(A) \to K_0(\hat{A})$ is an isomorphism.

Proof. Let (s, t) be a system of parameters of R and let R' denote the s-adic completion of R. Then the t-adic completion of R' is its completion as a local ring and equals $\hat{R}$ (this can be seen using Corollary 5 on p. 171 of [5]). It follows from faithful flatness of $R' \to \hat{R}$ and normality of $\hat{R}$ that R' is normal. Therefore, to prove the theorem, it suffices to prove the following: if $R \subset \tilde{R}$ are normal local domains of dimension 2 such that $R/sR \to \tilde{R}/s\tilde{R}$ is an isomorphism for some s in the maximal ideal of R, then the conlcusions of Theorem 4.2 hold with 'cap' replaced by 'tilde'.

Proof of (i). Let $\tilde{P}$ be a projective $\tilde{A}$-module where $\tilde{A} = \tilde{R}[X_1, \ldots, X_n]$. We have a patching diagram

$$\begin{array}{ccc} A & \longrightarrow & \tilde{A} \\ \big\downarrow & & \big\downarrow \\ A_s & \longrightarrow & \tilde{A}_s \end{array} \tag{4.3}$$

So, in view of 2.3, it suffices to show that $\tilde{P}_s$ is $\tilde{A}_s$-free. Since $\tilde{R}_s$ is a Dedekind domain, $\tilde{P}_s$ is isomorphic to $\tilde{A}_s \otimes_{\tilde{R}_s} P_0$ for some projective $\tilde{R}_s$-module P_0 ([8], Theorem 4′). But P_0 is nothing but $\tilde{P}_s/(X_1, \ldots, X_n)\tilde{P}_s$ and the latter is free, being a localization of the free $\tilde{R}$-module $\tilde{P}/(X_1, \ldots, X_n)\tilde{P}$.

Proof of (ii). Let P be a projective A-module of rank $r \geq 3$ such that $\tilde{A} \otimes_A P$ is $\tilde{A}$-free. We know by the argument given above, that P_s is A_s-free. Fix an isomorphism $\alpha' : \bar{P} \to \bar{A}^r$ with 'bar' denoting 'mod$(X_1, \ldots, X_n)$'. Choose isomorphisms $\alpha_1 : P_s \to A_s^r$ and $\alpha_2 : \tilde{A} \otimes_A P \to \tilde{A}^r$ such that $\bar{\alpha}_1 = \alpha'_s$ and $\bar{\alpha}_2 = 1 \otimes \alpha'$. To see that this can be done, let β denote some isomorphism $P_s \to A_s^r$. Lift the automorphism $\alpha'_s \bar{\beta}^{-1}$ of $\bar{A}_s^r$ to an automorphism γ of A_s^r and set $\alpha_1 = \gamma\beta$. Similarly for α_2.

The automorphism $(1 \otimes \alpha_1)(\alpha_2)_s^{-1}$ of $\tilde{A}_s^r$, viewed as a matrix with respect to the canonical basis, is of determinant 1 since it is identity mod$(X_1, \ldots, X_n)$; hence it belongs to $E_r(\tilde{A}_s)$ by Corollary 6.5 of [9] because $\tilde{R}_s$ is a Dedekind domain. Applying Remark 2.4 and Corollary 3.2 to the patching diagram (4.3) we conclude that $P \approx A^r$.

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